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LETTER TO THE EDITOR

R-matrix formulation of deformed boson algebra

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Abstract. Rewriting the Pusz–Woronowicz q-boson relations in terms of the standard R-matrix for $SU_q(n)$, we give the definition of a general deformed boson algebra $\mathcal{A}(R)$ depending upon a matrix R. We investigate the conditions under which this algebra is associative. For $n = 2$, a set of matrices satisfying these conditions is classified, and the corresponding 'twisted statistics' is given.

The interest in quantum groups [1,2] and quantum enveloping algebras [3] (see also [4–6] for introductions to these topics) has led to the study of q-deformations of the Heisenberg–Weyl algebra and the introduction of so-called q-bosons [7–9]. Two types of q-bosons for the quantum enveloping algebra $su_q(n)$, or the quantum group $SU_q(n)$, have been introduced. On the one hand, there are the Biedenharn–Macfarlane q-bosonic operators [7,8], which give rise to symmetric irreducible representations of $su_q(n)$ and to a Jordan–Schwinger realization of the $su_q(n)$ Chevalley generators [10]. On the other hand, there are the Pusz–Woronowicz q-boson operators [9], related to a covariant differential calculus on the quantum group $SU_q(n)$. Here, the creation operators transform as the components of the fundamental representation of $su_q(n)$, and the annihilation operators as the components of the dual representation. Also, the Pusz–Woronowicz operators transform covariantly under the action of the quantum group $SU_q(n)$. As operators acting in the q-Fock space, the Pusz–Woronowicz and the Biedenharn–Macfarlane operators can be related to each other [11–13].

In this letter, we shall define the deformed boson algebra in terms of the Pusz–Woronowicz operators. Their creation and annihilation operators are related by expressions involving the fundamental R-matrix of $SU_q(n)$. Taking these relations as the starting point for our definition of the deformed boson algebra, with an arbitrary matrix R, we investigate the conditions under which an associative algebra with Hermitian conjugate is obtained. Here, the technique is similar to that developed in [14, 15], but the ansatz is different. This leads to three conditions for the matrix R (a Hermiticity condition, the Yang–Baxter equation (YBE), and a Hecke condition), which are clearly satisfied by the $SU_q(n)$ fundamental R-matrix. For $n = 2$, a particular set of matrices R satisfying these conditions is classified, showing that apart from the trivial solution and the usual $SU_q(2)$ solution (R_q and its double), there is a third matrix R satisfying the conditions required here.

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The Pusz–Woronowicz q-bosonic creation and annihilation operators $A_i^\dagger$ and A_i ($i = 1, \ldots, n$) satisfy the commutation relations [9, 12, 13]

$$\begin{aligned} &A_i A_j - q A_j A_i = 0 \qquad i < j \\ &A_i^\dagger A_j^\dagger - q^{-1} A_j^\dagger A_i^\dagger = 0 \qquad i < j \\ &A_i A_j^\dagger - q A_j^\dagger A_i = 0 \qquad i \neq j \\ &A_i A_i^\dagger - q^2 A_i^\dagger A_i = 1 + (q^2 - 1) \sum_{j=1}^{i-1} A_j^\dagger A_j \,. \end{aligned} \tag{1}$$

These relations can be re-expressed in terms of the fundamental R-matrix of $SU_q(n)$. This is an $n^2 \times n^2$ matrix, and can be written as [6]

$$R = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji} \tag{2}$$

where e_{ij} is the $n \times n$ matrix with entry 1 at position (i, j) and 0 elsewhere. Note that

$$R = R_{ij,kl}\, e_{ik} \otimes e_{jl} \tag{3}$$

in (3), and in the rest of the letter there is summation over repeated indices. Putting $V = \mathbb{C}^n$, R can be seen as an element of $\mathrm{End}(V \otimes V)$. The twist operator $P \in \mathrm{End}(V \otimes V)$ is defined as $P(x \otimes y) = y \otimes x$, for all $x, y \in V$; in matrix notation we have $P_{ij,kl} = \delta_{il}\delta_{jk}$.

In terms of (2), the relations (1) can be rewritten as

$$A_i A_j = q^{-1} R_{ij,kl} A_l A_k \tag{4}$$

$$A_i^\dagger A_j^\dagger = q^{-1} R_{lk,ij} A_k^\dagger A_l^\dagger \tag{5}$$

$$A_i A_j^\dagger = \delta_{ij} + q R_{ki,jl} A_k^\dagger A_l \,. \tag{6}$$

This form of the q-bosonic relations can be deduced from the connection between the q-boson operators and the differential calculus for $SU_q(n)$ [9, 16–18]. In this letter, (4)–(6) will be the starting point for our definition of the deformed boson algebra $\mathcal{A}$. This will be a complex algebra generated by $\mathbb{C}$ and the elements $A_i^\dagger$, A_i ($i = 1, \ldots, n$), equipped with a Hermitian conjugation † which is an antihomomorphism (i.e. $(ab)^\dagger = b^\dagger a^\dagger$) such that $(A_i)^\dagger = A_i^\dagger$, $(A_i^\dagger)^\dagger = A_i$, and $(\lambda)^\dagger = \lambda^*$ (complex conjugate) for $\lambda \in \mathbb{C}$. Moreover, there will be quadratic relations similar to (4)–(6), but with q^{-1} and q replaced by independent complex numbers p and p'. Note that from the invariance under Hermitian conjugation of the relation

$$A_i A_j^\dagger = \delta_{ij} + p' R_{ki,jl} A_k^\dagger A_l \tag{7}$$

it follows that the matrix R must satisfy

$$R_{ij,kl} = R^*_{lk,ji} \,. \tag{8}$$

Thus, we have the following definition:

Definition. Let R be a complex $n^2 \times n^2$ matrix satisfying (8). The deformed boson algebra $\mathcal{A}(R)$ is the complex algebra generated by 1, $A_i^\dagger$ and A_i ($i = 1, \ldots, n$) subject to the relations

$$A_i A_j = pR_{ij,kl} A_l A_k \tag{9}$$

$$A_i^\dagger A_j^\dagger = p^* R^*_{ji,kl} A_k^\dagger A_l^\dagger \tag{10}$$

$$A_i A_j^\dagger = \delta_{ij} + p' R_{ki,jl} A_k^\dagger A_l \tag{11}$$

with antihomomorphism † satisfying $(A_i)^\dagger = A_i^\dagger$, $(A_i^\dagger)^\dagger = A_i$, and $(\lambda)^\dagger = \lambda^*$ ($\lambda \in \mathbb{C}$).

Note that (10) follows from applying † to (9). In this definition, p is a complex number, and p' is a real number.

Such an algebra would be of little interest if it were not associative. In the following, we shall investigate the conditions under which $\mathcal{A}(R)$ is an associative algebra. To express associativity, it is sufficient to require the braid transposition schemes for triples of generators of $\mathcal{A}(R)$ [2]. This is by now a well known technique. For the product $A_i A_j A_k$, the braid transposition scheme is

$$\left(\begin{array}{ccccc} & & ikj \;\rightarrow\; kij & & \\ & \nearrow & & \searrow & \\ ijk & & & & kji \\ & \searrow & & \nearrow & \\ & & jik \;\rightarrow\; jki & & \end{array} \right). \tag{12}$$

Applying this to $A_i A_j A_k$, using (9), yields the YBE for the matrix R:

$$\sum_{u,v,w} R_{ab,uv} R_{vw,cd} R_{ue,fw} = \sum_{u,v,w} R_{be,uv} R_{wu,fc} R_{av,wd} \, . \tag{13}$$

This relation between the braid scheme and the YBE is known; it also follows from the fact that the A_i satisfy the same relation (9) as the quantum plane coordinates [2], and the equivalence of the associativity of the quantum plane coordinates and the YBE for R.

Similarly, the braid scheme for $A_i^\dagger A_j^\dagger A_k^\dagger$ gives rise to the YBE equation for R^* or, using (8), to the YBE for R. The triples of generators that remain to be investigated are $A_i A_j A_k^\dagger$ and $A_i A_j^\dagger A_k^\dagger$. Here, the calculations are similar as in [14,15], but the starting point is different. We shall give one calculation in more detail here. Using the top half of (12) on $A_i A_j A_k^\dagger$, i.e. first use (11) on the last two components, then (11) on the first two components, and finally (9) on the last two components, one finds

$$A_i A_j A_k^\dagger = \delta_{jk} A_i + p' R_{ij,kb} A_b + p'^2 p R_{aj,kb} R_{ui,av} R_{vb,xy} A_u^\dagger A_y A_x \, . \tag{14}$$

Using the bottom half of (12) on $A_i A_j A_k^\dagger$, one finds

$$A_i A_j A_k^\dagger = pR_{ij,kb} A_b + pp' R_{ij,au} R_{ua,kv} A_v + p'^2 p R_{ij,ab} R_{ua,kv} R_{xb,uy} A_x^\dagger A_y A_v \, . \tag{15}$$

The cubic terms in the right-hand sides of (14) and (15) are equal if and only if R satisfies the YBE. The linear terms are equal if the following relation is satisfied:

$$pp'R_{ij,ab}R_{ba,kl} + pR_{ij,kl} - p'R_{ij,kl} - \delta_{jk}\delta_{il} = 0 . \tag{16}$$

Using the notation $\check{R} = PR$, this can be rewritten as follows:

$$(p\check{R} - 1)(p'\check{R} + 1) = 0 . \tag{17}$$

This is the Hecke condition for $\check{R}$, and implies that $\check{R}$ has two eigenvalues p^{-1} and $-p'^{-1}$. Similarly, if one considers the braid scheme for $A_i A_j^\dagger A_k^\dagger$, using (8), one can show that the cubic terms again give rise to the YBE for R, and that the linear terms give the condition

$$(p^*\check{R} - 1)(p'\check{R} + 1) = 0. \tag{18}$$

It follows that p has to be real. Thus we have the following result:

Theorem. The deformed boson algebra $\mathcal{A}(R)$ with p and p' real is an associative algebra provided R satisfies the YBE (13) and the Hecke condition (17).

The following is a classical remark concerning the quantum group and is worth repeating here in terms of the deformed boson algebra relations. Consider the following transformations for the deformed bosons:

$$B_i = M_{ij}A_j \qquad B_i^\dagger = N_{ji}A_j^\dagger \tag{19}$$

where the elements M_{ij} and N_{ji} are supposed to commute with A_k and $A_k^\dagger$, but not among themselves. Then (9) holds for the B-operators provided

$$RM_2M_1 = M_1M_2R \tag{20}$$

where $M_1 = M \otimes 1$ and $M_2 = 1 \otimes M$. Similarly, (10) holds for the $B^\dagger$-operators provided

$$RN_1N_2 = N_2N_1R . \tag{21}$$

Finally, relation (11) is valid for the $\{B, B^\dagger\}$-operators if

$$M_{ia}N_{aj} = \delta_{ij} \qquad N_{ia}M_{aj} = \delta_{ij} . \tag{22}$$

If this last relation holds, then (21) and (20) are equivalent statements.

Let us now turn to the study of matrices R satisfying the three properties (8), (13) and (17) in the case $n = 2$. It would still be a formidable task to find all matrices satisfying these three conditions. Therefore, we make one further assumption. We shall assume in (9) that in the relations between A_1A_2 and A_2A_1 no A_1A_1 and A_2A_2 appear, and vice versa. Concretely, this means that the matrix R now takes the special form

$$\begin{pmatrix} \times & 0 & 0 & \times \\ 0 & \times & \times & 0 \\ 0 & \times & \times & 0 \\ \times & 0 & 0 & \times \end{pmatrix} .$$

Let us choose the following labelling for R (p is real):

$$pR = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c' & b^* & 0 \\ d^* & 0 & 0 & a' \end{pmatrix}$$

where it follows from (8) that $a, a', c, c' \in \mathbb{R}$ and $b, d \in \mathbb{C}$. Putting $B = pPR$ and using (17), the matrix B must satisfy $(B-1)(B+\alpha) = 0$, with $\alpha = p/p'$. This leads to the following conditions:

$$\begin{aligned} &(a + a' + \alpha - 1)d = 0 \qquad && dd^* = (a+\alpha)(1-a) = (a'+\alpha)(1-a') \\ &(c + c' + \alpha - 1)b = 0 \qquad && bb^* = (c+\alpha)(1-c) = (c'+\alpha)(1-c')\,. \end{aligned} \tag{23}$$

Finally, there are the conditions following from the YBE for R (or for pR); these are rather numerous and we will not write them here explicitly. It is, however, possible to solve the system of equations completely. The following solutions emerge:

(i) $d = 0$, $b = 0$.
The YBE leads to $a = a' = c = c' = 1$ or $a = a' = c = c' = -\alpha$. Thus the solutions are

$$pR = P \qquad \text{or} \qquad pR = -\alpha P\,.$$

(ii) $d = 0$, $b \neq 0$.
The YBE implies $cc' = 0$. There are two cases to distinguish. If $c = 0$, then $c' = 1-\alpha$ and $bb^* = \alpha$. Thus α must be positive. Putting $\alpha = q^2$ (q real), the most general solution in this case is:

$$pR = \begin{pmatrix} \{1,-q^2\} & 0 & 0 & 0 \\ 0 & \pm q & 0 & 0 \\ 0 & 1-q^2 & \pm q & 0 \\ 0 & 0 & 0 & \{1,-q^2\} \end{pmatrix} \tag{24}$$

where $\{1, -q^2\}$ indicates that for this entry one can choose either 1 or else $-q^2$. In the second case, $c' = 0$ and $c = 1 - \alpha$. Using the same notation, this leads to

$$pR = \begin{pmatrix} \{1,-q^2\} & 0 & 0 & 0 \\ 0 & \pm q & 1-q^2 & 0 \\ 0 & 0 & \pm q & 0 \\ 0 & 0 & 0 & \{1,-q^2\} \end{pmatrix}\,. \tag{25}$$

The classical R-matrix, and its quantum double, belong to this class of solutions.

(iii) $d \neq 0$.
In this case, the YBE implies $c = c' = (1-\alpha)/2$. The last equation of (23) implies $bb^* = ((\alpha+1)/2)^2$, and the YBE implies that $dd^* = cc' = ((1-\alpha)/2)^2$. From a number of conditions following from the explicit form of the YBE, one deduces that

$\alpha \geqslant 0$, so again we put $\alpha = q^2$; then $a = (1-q^2)/2 \pm q$ and $a' = (1-q^2)/2 \mp q$, and b and d must be real. The most general solution in this case reads

$$pR = \begin{pmatrix} \frac{1-q^2}{2} + \epsilon q & 0 & 0 & \epsilon' \frac{1-q^2}{2} \\ 0 & \epsilon'' \frac{1+q^2}{2} & \frac{1-q^2}{2} & 0 \\ 0 & \frac{1-q^2}{2} & \epsilon'' \frac{1+q^2}{2} & 0 \\ \epsilon' \frac{1-q^2}{2} & 0 & 0 & \frac{1-q^2}{2} - \epsilon q \end{pmatrix} . \tag{26}$$

Here, ϵ, ϵ' and ϵ'' are three independent signs: $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$.

Let us now consider explicitly the 'statistics' of the deformed boson operators in these situations. For case (i), the relations are easy to work out and rather trivial. For case (ii), (24) and (25) are similar; here, we only consider (25). Note that the matrix (25) can in its most general form be rewritten as follows:

$$pR = \begin{pmatrix} \frac{1-q^2}{2} + \epsilon \frac{1+q^2}{2} & 0 & 0 & 0 \\ 0 & \epsilon' q & 1-q^2 & 0 \\ 0 & 0 & \epsilon' q & 0 \\ 0 & 0 & 0 & \frac{1-q^2}{2} + \epsilon'' \frac{1+q^2}{2} \end{pmatrix} \tag{27}$$

with three independent signs: $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$. The relations following from (9) read

$$(\epsilon - 1) A_1 A_1 = (\epsilon'' - 1) A_2 A_2 = 0 \qquad A_1 A_2 = \epsilon' q^{-1} A_2 A_1 . \tag{28}$$

The relations among $A_i^\dagger$ follow by applying the antihomomorphism, and the relations (11) become

$$\begin{aligned} & A_1 A_1^\dagger - \left(\frac{q^{-2}-1}{2} + \epsilon \frac{q^{-2}+1}{2} \right) A_1^\dagger A_1 = 1 \\ & A_i A_j^\dagger = \epsilon' q^{-1} A_j^\dagger A_i \qquad i \neq j \\ & A_2 A_2^\dagger - \left(\frac{q^{-2}-1}{2} + \epsilon'' \frac{q^{-2}+1}{2} \right) A_2^\dagger A_2 = 1 + (q^{-2} - 1) A_1^\dagger A_1 . \end{aligned} \tag{29}$$

With $\epsilon = \epsilon' = \epsilon'' = 1$ and q replaced by q^{-1}, these relations coincide with (1). When $q = 1$ in (28) and (29), this becomes

$$\begin{aligned} & (\epsilon - 1) A_1^2 = 0 && A_1 A_1^\dagger - \epsilon A_1^\dagger A_1 = 1 \\ & A_1 A_2 - \epsilon' A_2 A_1 = 0 && A_i A_j^\dagger - \epsilon' q^{-1} A_j^\dagger A_i = 0 \qquad i \neq j \\ & (\epsilon'' - 1) A_2^2 = 0 && A_2 A_2^\dagger - \epsilon'' A_2^\dagger A_2 = 1 . \end{aligned} \tag{30}$$

For $\epsilon = 1$ (resp. -1), $(A_1, A_1^\dagger)$ is a boson (resp. a fermion) annihilation and creation operator pair. Similarly, for $\epsilon'' = 1$ (resp. -1), $(A_2, A_2^\dagger)$ is a boson (resp. a fermion) pair. For $\epsilon' = 1$ (resp. -1), the two modes commute (resp. anticommute).

Finally, consider the peculiar statistics implied by case (iii). From (9), the following two relations are obtained:

$$(1-\epsilon q)A_1A_1 = \epsilon'(1+\epsilon q)A_2A_2 \qquad A_1A_2 = \epsilon'' A_2A_1 \tag{31}$$

and similarly for the quadratic relations in $A_i^\dagger$. From (11), one finds

$$\begin{aligned}
A_1A_1^\dagger + A_1^\dagger A_1 &= 1 + \frac{\epsilon q^{-1}+1}{2}((\epsilon q^{-1}+1)A_1^\dagger A_1 + (\epsilon q^{-1}-1)A_2^\dagger A_2) \\
A_1A_2^\dagger - \epsilon'' A_2^\dagger A_1 &= \frac{q^{-2}-1}{2}(\epsilon' A_1^\dagger A_2 + \epsilon'' A_2^\dagger A_1) \\
A_2A_1^\dagger - \epsilon'' A_1^\dagger A_2 &= \frac{q^{-2}-1}{2}(\epsilon' A_2^\dagger A_1 + \epsilon'' A_1^\dagger A_2) \\
A_2A_2^\dagger + A_2^\dagger A_2 &= 1 + \frac{\epsilon q^{-1}-1}{2}((\epsilon q^{-1}+1)A_1^\dagger A_1 + (\epsilon q^{-1}-1)A_2^\dagger A_2)\,.
\end{aligned} \tag{32}$$

When $q = 1$ in (31) and (32), one can verify that it is a system of one boson pair and one fermion pair which commute or anticommute (depending on whether ϵ'' is 1 or -1).

The relations (28), (29) and (31), (32) can be considered as a further generalization of the Pusz–Woronowicz relations (1) in terms of an R-matrix which is still compatible with an associative algebra. All the classical situations, such as a system of two bosons or of two fermions, are easily seen to be special limits of the above deformed cases.

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